Linear Algebra Problems

12. Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and define the set $W := U + V$ as the set of all vectors $w = u + v$ where $u \in U$ and $v \in V$ can be any vectors.

   a) Show that W is a linear space.

   b) Find all possible values for the dimension of W.

a) To show that W is a linear space.

where W is set of all vectors, w = u + v where u ∈ U and v ∈ V can be any vectors and U and V are two-dimensional subspaces of R^5

that is to show 1) if $w_1, w_2 \in W$ then $w_1 + w_2 \in W$

2) w ∈ W then kw ∈ W where k is a scalar

3) 0 ∈ W

1) let $w_1 = u_1 + v_1$ and $w_2 = u_2 + v_2$ be two vectors in W, where u_1 and u_2 are in U and v_1 and v_2 are in V then $w_1 + w_2 = (u_1 + u_2) + (v_1 + v_2)$ where $u_1 + u_2 \in U$ and $v_1 + v_2 \in V$ and hence $w_1 + w_2 \in W$

2) let w= u + v where u ∈ U and v ∈ V, then kw = ku+kv where ku ∈ U and kv ∈ V hence kw ∈ W

3) 0= u+ (-u) where u ∈ U and (-u) ∈ V hence 0 ∈ W

b) To find all possible values for the dimension of W

Dimension of W may be 0,1,2,3 and 4

$[x_1, x_2, 0, 0, 0] + [-x_1, -x_2, 0, 0, 0]$ has dimension 0.

$[x_1, x_2, 0, 0, 0] + [-x_1, x_3, 0, 0, 0]$ has dimension 1.

$[x_1, x_2, 0, 0, 0] + [x_3, x_4, 0, 0, 0]$ has dimension 2.

$[x_1, x_2, 0, 0, 0] + [0, x_3, x_4, 0, 0]$ has dimension 3.

$[x_1, x_2, 0, 0, 0] + [0, 0, x_3, x_4, 0]$ has dimension 4.

18)

18. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that the following are equivalent.

a) A is surjective (hence $n \geq k$).

b) $\dim \operatorname{im}(A) = k$.

c) A has a *right inverse* B, so $AB = I$.

d) The columns of A span $\mathbb{R}^k$.

I will show (a) => (b) => (c) => (d) => (a).

(a) => (b): Assume A is surjective. Then the image of A is all of l k , which has dimension k.

(b) => (c) : Since the dimension of im(A) is k, every vector in R^k is in the image of A. Choose a basis [f1,f2, . . . ,fk] of R^k , and then choose vectors e1, . . . , ek in R^n such that A(e1) = f1, A(e2) = f2,. . . ,A(ek) = fk. Set B(f1) = e1, B(f2) = e2,. . . ,B(fk) = ek. Then B is a right inverse for A. so AB=I

(c) => (d) : If A has a right inverse, then for every vector f $\in R^k$, we have A(B(f)) = f. Thus every vector f $\in R^k$ is in the image of A, which is the span of the columns of A. Thus the columns of A span R^k.

(d) => (a): The span of the columns of A is the image of A. So if the columns of A span all of R^k , then every f $\in$ R^k is a linear combination of the columns of A, and the n coefficients in this linear combination are the components of a vector e $\in R^n$ with the property that A(e) =f. Thus A is surjective

26)

26. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees ($+45$ degrees means 45 degrees *counterclockwise*).

b) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

c) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

e) Find the inverse of each of these maps.

a)

This rotation sends the vector (1, 0) to $(1/\sqrt{2}, 1/\sqrt{2})$ and sends the vector (0, 1) to $(-1/\sqrt{2}, 1/\sqrt{2})$. We put the image vectors in columns and get the matrix.

$T_{45} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

b) Reflection across the horizontal (x) axis sends (1, 0) to itself and sends (0, 1) to (0, −1) and so its matrix is

$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Therefore rotation followed by reflection is given by the matrix

$FT_{45} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

(c) On the other hand, reflection followed by rotation is given by

$T_{45}F = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

d) Rotation by 60 degrees sends (1, 0) to $(1/2, \sqrt{3}/2)$ and sends (0, 1) to $(-\sqrt{3}/2, 1/2)$. Thus

$T_{60} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

e) The inverse of T_{45} is T_{-45} which is nothing but $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Reflection in the x axis is its own inverse i.e. $F^{-1}=F$

$(FT_{45})^{-1} = T_{45}^{-1}F^{-1} = T_{-45}F = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Likewise

$(T_{45}F)^{-1} = F^{-1}\, T_{45}^{-1} = F\, T_{-45} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

And finally $T_{60}^{-1} = T_{-60} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$

63)

63. Let $\mathcal{S}$ be the linear space of infinite sequences of real numbers $x := (x_1, x_2, \ldots)$. Define the linear map $L : \mathcal{S} \to \mathcal{S}$ by

$$Lx := (x_1 + x_2,\ x_2 + x_3,\ x_3 + x_4, \ldots).$$

a) Find a basis for the nullspace of L. What is its dimension?
b) What is the image of L? Justify your assertion.
c) Compute the eigenvalues of L and an eigenvector corresponding to each eigenvalue.

(a) If x = (x1, x2, . . .) ∈ null space of L then x1+ x2 = 0, x2+ x3= 0,. . . . So if x1 = a, we must have x2 = −a so that x1 +x2 = 0, then x3 = a so that x2 +x3 = 0 and so forth. So the null space is one-dimensional, and a basis is the infinite vector (1, −1, 1, −1, . . .).

b) The image is all of S, since if b = (b1, b2, . . .) is any sequence, we can set x1 = 0 and then let x2 = b1, x3 = b2 − b1, x4 = b3 − b2 +b1, x5 = b4 − b3+ b2 − b1,. . . and then for x = (x1, x2, x3, . . .) we will have Lx = b.

c) If λ is an eigenvalue of L, then its eigenvector x = (x1, x2, x3, . . .) satisfies x1+ x2 = λx1,

x2+ x3=λx2,.. . ….

Solving this will give x2=(λ-1)*x1, x3=(λ-1)2*x1,……. Xn=$(\lambda-1)^{n-1}$*x1,…………

So every number λ ∈ R is an eigenvalue and the corresponding eigenvector is given as (let x1=1)

x = $(1, (\lambda - 1), (\lambda - 1)^2, (\lambda - 1)^3, \ldots)$.

76) a) tr(AB)=$\sum_{i=1}^{N} (AB)ii = \sum_{i=1}^{N} \sum_{j=1}^{N} AijBji$

=$\sum_{j=1}^{N} \sum_{i=1}^{N} Bji\, Aij$ (By using the fact that iand j are dummy summation indices with the same range)

=$\sum_{j=1}^{N} (AB)jj$

=tr(BA)

b)

tr(ABC)== $\sum_i (ABC)ii = \sum_{ijk} Aij\ Bjk\ Cki$

By using the fact that i, j, and k are dummy summation indices with the same range, this sum can be written in the equivalent forms

$\sum_{ijk} Aij\ Bjk\ Cki = \sum_{ijk} Cki\ Aij\ Bjk\ = \sum_{ijk}\ Bjk\ Cki\ Aij$

But the second and third of these are

$$\sum_{ijk} Cki\ Aij\ Bjk\ = tr(CAB)$$

and $\sum_{ijk}\ Bjk\ Cki\ Aij = tr(BCA)$ respectively.

Thus, we obtain the relation

tr(ABC) = tr(CAB) = tr(BCA)

c)

NO

Counter example:

A=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

B=$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

C=$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Here tr(ABC)=3 but tr(BAC)=2 (ABC=$\begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}, BAC = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$)

80)

80. a) Find a 2×2 real matrix A that has an eigenvalue $\lambda_1 = 1$ with eigenvector $E_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and an eigenvalue $\lambda_2 = -1$ with eigenvector $E_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b) Compute the determinant of $A^{10} + A$.

Let matrix A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then according to question for λ_1=1 and E_1=$\frac{1}{2}$

$AE_1 = \lambda_1 E_1$

⇨ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} = \begin{matrix} 1 \\ 2 \end{matrix}$

⇨ a+2b=1

and c+2d=2

Again for λ_2=-1 and $E_2=\begin{matrix} 2 \\ 1 \end{matrix}$

$AE_1= \lambda_1 E_1$

⇨ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{matrix} 2 \\ 1 \end{matrix} = \begin{matrix} -2 \\ -1 \end{matrix}$

⇨ $2a + b = -2$

and 2c+ d= -1

Solving these four equation gives a=$-5/3$, b=$4/3$,c=$-4/3$, d=$5/3$

So A=$\begin{bmatrix} -5/3 & 4/3 \\ -4/3 & 5/3 \end{bmatrix}$

A^2=A*A=$\begin{bmatrix} -5/3 & 4/3 \\ -4/3 & 5/3 \end{bmatrix} * \begin{bmatrix} -5/3 & 4/3 \\ -4/3 & 5/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ =I where I is identity matrix

So $A^{10}=(A^2)^5=I^5=I$

So A^{10}+A=I+A=$\begin{bmatrix} -2/3 & 4/3 \\ -4/3 & 8/3 \end{bmatrix}$

Hence its determinant will be 0.

95)

95. Let L be an $n \times n$ matrix with real entries and let λ be an eigenvalue of L. In the following list, identify all the assertions that are correct.

a) $a\lambda$ is an eigenvalue of aL for any scalar a.

b) λ^2 is an eigenvalue of L^2.

c) $\lambda^2 + a\lambda + b$ is an eigenvalue of $L^2 + aL + bI_n$ for all real scalars a and b.

d) If $\lambda = a + ib$, with $a,\ b \neq 0$ some real numbers, is an eigenvalue of L, then $\bar{\lambda} = a - ib$ is also an eigenvalue of L.

λ be an eigenvalue of L hence Lx= λx ---------------- 1

a) multiply by scalar a in equation 1, we get aLx=(aλ)x hence $a\lambda$ will be eigenvalue of aL

b) multiply by matrix L in equation 1, we get $LLx= \lambda Lx$

i.e. $L^2x= \lambda(\lambda x)= \lambda^2x$

hence λ^2 will be eigenvalue of L^2

c) consider the matrix, $L^2+aL+bI_n$

Now $(L^2+aL+bI_n)x = L^2x + aLx + bI_nx$

$= \lambda^2x + a\lambda x +bx$ (from part b and part a we have $L^2x= \lambda^2x$ and $aLx=(a\lambda)x$)

$= (\lambda^2 + a\lambda +b)x$

Hence $\lambda^2+ a\lambda + b$ is an eigenvalue of $L^2+ aL + bI_n$ for all real scalars a and b .

d) We know that if λ is eigenvalue of a matrix A (nxn) then λ will satisfy $Ax=\lambda x$

That is to find an eigenvalue we need to solve $(A-\lambda I_n)x=0$

That is determinant of $A-\lambda I_n$ equals to zero

Hence we will get n degree equation in λ and its all the n roots will be the eigenvalues. But as we know that complex root always occur in conjugate pair hence if $\lambda = a + ib$ is root then its conjugate $\lambda' = a - ib$ will also be a root hence

If $\lambda = a + ib$, with $a, b \neq 0$ some real numbers, is an eigenvalue of A , then $\lambda' = a - ib$ is also an eigenvalue of A

---THE END--